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DECOMPOSITION TECHNIQUES FOR FINITE SEMIGROUPS, USING CATEGORIES I

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Dans cet article, nous donnons une classification complète des morphismes de semigroupes non factorisables $\theta : S \to T$, où S et T sont finis. Certaines propriétés des morphismes de semigroupes finis $\varphi : S \to T$ sont mises en relation avec les classes des morphismes non factorisables dont φ est le produit.

In this paper, we give a complete classification of non-factorizable semigroup morphisms $\theta : S \to T$, where S and T are finite. Certain properties of semigroup morphisms $\varphi : S \to T$ are put in correspondance with the classes of the non-factorizable morphisms of which φ is the product.

Introduction

All the semigroups considered here are finite. In this paper, we give a complete classification of non-factorizable semigroup morphisms $\theta : S \to T$.

The motivation for this work will be made explicit in part II [8], in which we shall characterize, for each of the classes that we distinguish here, classes of monoids V such that there exists an injective relational morphism (division) $\varphi : S < V \wr T$ for which $\theta = \varphi\pi$. Here $\wr$ denotes either the semidirect or the 2-sided semidirect product, and π denotes the canonical projection of $V \wr T$ onto T. These results will then be extended to larger classes of relational morphisms θ and in particular to aperiodic morphisms, **LG**- and **LI**-morphisms and regular **LG**- and **LI**-morphisms. The results of this article and part II were announced in [7]. They are also in [10].

The classification of non-factorizable semigroup morphisms, or maximal proper surmorphisms (m.p.s.'s) was first studied in [5] in which one can find a classification somewhat rougher than the one presented here.

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The paper is divided as follows: Section 1 reminds the reader of the classical results of semigroup theory that will be used, and in particular of facts relative to the Schützenberger group of a $\mathcal{J}$-class, and to the varieties of finite semigroups, monoids and categories. In Section 2, the definition and general properties of m.p.s.'s are discussed. Finally, the detailed classification of m.p.s.'s is described in Section 3, where numerous examples are given.

1. Preliminaries

We shall review in this section some classical tools for the description of semigroups and relational morphisms. See [2] and [9]. Recall that, if S and T are semigroups, a *relational morphism* $\varphi : S \to T$ is a relation (i.e. a mapping from S into 2^T) such that, for all s and t in S, $s\varphi \neq \emptyset$ and $(s\varphi)(t\varphi) \subseteq (st)\varphi$. φ is *injective* if, further, $s\varphi \cap t\varphi \neq \emptyset$ implies $s=t$. In this case, we say that S *divides* T and we write $\varphi : S < T$ (or $S < T$).

We denote by S^I the monoid $S \cup \{I\}$ where I is an identity and by $S^{\cdot}$ the monoid equal to S if S is a monoid, to S^I otherwise. $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{J}$ and $\mathcal{D}$ denote the classical Green relations.

1.1. Schützenberger group of a $\mathcal{J}$-class

We recall here a few results concerning the structure of the $\mathcal{J}$-classes of a semigroup S. For detailed proofs, see [1, 2, 4, 10].

Let A and B be finite non-empty sets, G be a finite group, and P be a $B \times A$-matrix with entries in $G \cup \{0\}$. $\mathcal{M}^0(A, B, G, P)$ denotes the semigroup $(A \times G \times B) \cup \{0\}$ defined by $(a, g, b)(a', g', b') = (a, gp_{b,a'}g', b')$, if $p_{b,a'} \neq 0$, 0 otherwise. $\mathcal{M}^0(A, B, G, P)$ is regular iff each row and each column of P has a non-zero entry; in this case it is a 0-simple semigroup and $A \times G \times B$ is a $\mathcal{J}$-class.

If J is a $\mathcal{J}$-class of a semigroup S, Let $J^0 = J \cup \{0\}$ be the semigroup defined, for all s and t in J, by $s \cdot t = st$ if $st \in J$, 0 otherwise. It is well known that $J^0 = \mathcal{M}^0(A, B, G, P)$ for some A, B, G, P, with P either identically zero or a regular matrix. Moreover, the sets A and B can be chosen to be respectively the sets of $\mathcal{R}$- and $\mathcal{L}$-classes of J. In fact, the $\mathcal{R}$-classes are $R_a^S = \{a\} \times G \times B$ $(a \in A)$ and the $\mathcal{L}$-classes are $L_b^S = A \times G \times \{b\}$ $(b \in B)$. The group G is the *Schützenberger group* of J. Given any $\mathcal{H}$-class H of J, G is isomorphic to the groups of permutations of H induced, respectively, by the right and left translations by elements of S that preserve H. Both these groups are regular transitive permutation groups whose actions commute. If H is itself a group, then G is isomorphic to H.

Let then $\theta : S \to T$ be a morphism, J be a $\mathcal{J}$-class of S, and J' be the $\mathcal{J}$-class of T containing $J\theta$. $J^0 = \mathcal{M}^0(A, B, G, P)$ and $J'^0 = \mathcal{M}^0(A', B', G', P')$ for some A, A', B, B', G, G', P, P'. Since $\mathcal{R}$- (resp. $\mathcal{L}$-) equivalent elements of S have $\mathcal{R}$- (resp. $\mathcal{L}$-) equivalent images under θ, θ induces mappings $\alpha : A \to A'$ and $\beta : B \to B'$ such that $R_a^S \theta \subseteq R_{a\alpha}^T$ and $L_b^S \theta \subseteq L_{b\beta}^T$ for all $a \in A$ and $b \in B$. Let us note that, if $p_{b,a} \neq 0$

$(a\in A, b\in B)$, then $L_b^S\cdot R_a^S\subseteq J$ and hence $L_{b\beta}^T\cdot R_a^S\subseteq J$ and hence $L_{b\beta}^T\cdot R_{a\alpha}^T\subseteq J'$, i.e. $p'_{b\beta,a\alpha}\neq 0$. Finally, the characterizations of G and G' as permutation groups over arbitrary $\mathscr{H}$-classes of J and J' make it easy to check that θ induces a group morphism $\tilde{\theta}: G\to G'$.

Since we shall need later a more precise description of the action of θ on J, we now turn to (somewhat painful) technicalities. Let us first choose reference elements, denoted by 1, in A and B, and let us denote also by 1 the elements 1α and 1β in A' and B'. For each $a\in A$ (resp. $b\in B$) there exist u_a and $\bar{u}_a$ (resp. v_b and $\bar{v}_b$) in $S^{\cdot}$ such that the left translation by u_a (resp. the right translation by v_b) is a bijection from R_1^S onto R_a^S (resp. L_1^S onto L_b^S), and the left translation by $\bar{u}_a$ (resp. the right translation by $\bar{v}_b$) is the reciprocal bijection from R_a^S onto R_1^S (resp. L_b^S onto L_1^S). We choose u_1, $\bar{u}_1$, v_1 and $\bar{v}_1$ to be equal to 1. In fact, the isomorphism between $\mathscr{M}^0(A,B,G,P)$ and J^0 is given by assigning to (a,g,b) the element $u_a(h_0\cdot g)v_b$ where h_0 is a (arbitrarily) fixed element of the $\mathscr{H}$-class $H_{1,1}^S=R_1^S\cap L_1^S$ (here, G is identified with the permutation group of $H_{1,1}^S$ induced by the right translations of S that preserve $H_{1,1}^S$.

The same is true for the $\mathscr{J}$-class J' of T and we may choose, similarly, elements $u'_{a'}$, $\overline{u'_{a'}}$, $v'_{b'}$, $\overline{v'_{b'}}$ in $T^{\cdot}$ for all $a'\in A'$, $b'\in B'$, and h'_0 in $H_{1,1}^T=R_1^T\cap L_1^T$. We identify G' with the permutation group of $H_{1,1}^T$ induced by the right translations of T that preserve $H_{1,1}^T$. In particular, the group morphism $\tilde{\theta}$ is such that $g\tilde{\theta}$ $(g\in G)$ is the (uniquely determined) element of G' such that $(h_0\cdot g)\theta=h'_0\cdot g\tilde{\theta}$.

Let then $(a,g,b)=u_a(h_0\cdot g)v_b$ be any element of J. Then

$$(a,g,b)\theta=(u_a(h_0\cdot g)v_b)\theta=(u_a\theta)(h'_0\cdot g\tilde{\theta})(v_b\theta)$$

and, since $(a,g,b)\theta\in R_{a\alpha}^T\cap L_{b\beta}^T$,

$$(a,g,b)\theta=u'_{a\alpha}((\overline{u'_{a\alpha}}(u_a\theta))(h'_0\cdot g\tilde{\theta})((v_b\theta)\overline{u'_{b\beta}}))v'_{b\beta}.$$

Let us denote by d_b the element of G' induced by the right translation by $(v_b\theta)\overline{u'_{b\beta}}$, and by g_a the left permutation of $H_{1,1}^T$ induced by the left translation by $\overline{u'_{a\alpha}}(u_a\theta)$. Then, $(a,g,b)\theta=u'_{a\alpha}(g_a\cdot h'_0\cdot g\tilde{\theta}\cdot d_b)v'_{b\beta}$. See [1, Chapter 7] for more details.

1.2. Varieties and V-morphisms

Recall that an **S**- (resp. **M**-, **G**-) *variety* is a class of *finite* semigroups (resp. monoids, groups) closed under division and *finite* direct product. Varieties are studied in detail in numerous works, and in particular in [2,4]. We shall list here some useful varieties.

G is the variety of all groups and **I** the trivial **M**-variety.

A is the **M**-variety of aperiodic (i.e. combinatorial, group-free, $\mathscr{H}$-trivial) monoids.

R (resp. $\mathbf{R}^r$) is the *M*-variety of $\mathscr{R}$-trivial (resp. $\mathscr{L}$-trivial) monoids.

$\mathbf{J}_1$ is the variety of idempotent commutative monoids. It is generated by $U_1=\{1,0\}$.

If **V** is an **M**-variety, $\mathbf{V_S}$ is the **S**-variety generated by **V** and **LV** is the **S**-variety of all semigroups S such that, for any idempotent e of $\mathscr{S}$, eSe is in **V**. Note that $S \in \mathbf{LG}$ iff S has only one regular $\mathscr{J}$-class: its minimal ideal. Note also that **LI** is the **S**-variety of semigroups that are in **LG** and are aperiodic, i.e. $\mathbf{LI} = \mathbf{LG} \cap \mathbf{A_S}$

Let **V** be a **S**-variety and $\tau : S \to T$ be a relational morphism. τ is a **V**-*morphism* if, for any subsemigroup T' of T that is in **V**, $T'\tau^{-1}$ is also in **V**. If $\mathbf{V} = \mathbf{A_S}$, τ is said to be *aperiodic*. The following easy results make the determination of aperiodic and **LG**-morphisms easier. For a proof, see [4, 6, 9].

Proposition 1.1. *Let $\tau : S \to T$ be a relational morphism.*
- *The following are equivalent*:
  (1) *τ is aperiodic*;
  (2) *for any idempotent e of T, $er^{-1} \in \mathbf{A_S}$*;
  (3) *the restriction of τ to any group of S is injective.*
- *The following are equivalent*:
  (1) *τ is an* **LG**-*morphism*;
  (2) *for any idempotent e of T, $er^{-1} \in \mathbf{LG}$*;
  (3) *the restriction of τ to any copy of U_1 is injective.* □

LG-morphisms, when functional, are sometimes called $\gamma(U_1)$- or $\mathscr{J}'$-morphisms [1, 5, 6, 9].

Note also the following result:

Proposition 1.2. *Let $\beta_1 : S \to T$ and $\beta_2 : T \to V$ be surmorphisms, $\beta = \beta_1\beta_2$, and* **W** *be a* **S**-*variety. Then, β is a* **W**-*morphism iff β_1 and β_2 are* **W**-*morphisms.*

Proof. Let us assume first that β_1 and β_2 are **W**-morphisms and let us consider a subsemigroup V' of V that is in **W**. Then $V'\beta_2^{-1}$ is in **W**, and hence so is $V'\beta^{-1} = (V'\beta_2^{-1})\beta_1^{-1}$: β is a **W**-morphism. Conversely, let us assume that β is a **W**-morphism. If V' is a subsemigroup of V in **W**, then $V'\beta_2^{-1} = (V'\beta^{-1})\beta_1$ divides $V'\beta^{-1}$ and hence is in **W**: β_2 is a **W**-morphism. If T' is a subsemigroup of T that is in **W**, $T'\beta_1^{-1} \subseteq (T'\beta_2)\beta^{-1}$. Since $T'\beta_2 < T'$, $T'\beta_2$ is in **W** and hence so are $(T'\beta_2)\beta^{-1}$ and $T'\beta_1^{-1}$: β_1 is also a **W**-morphism. □

2. Maximal proper surmorphisms and θ-singular $\mathscr{J}$-classes

A *maximal proper surmorphism* or *m.p.s.* [5] is an onto semigroup morphism $\theta : S \to T$ that is not an isomorphism and is such that, if $\theta = \theta_1\theta_2$, then one of θ_1 and θ_2 is an isomorphism. It is clear that any onto morphism θ can be written as a product $\theta = \theta_1\theta_2 \cdots \theta_k$ of m.p.s.'s.

M.p.s.'s have been studied by the first author and this section reviews some of the results proved in [5]. In Section 3 we shall extend the results of [5] and classify the m.p.s.'s.

All the semigroup morphisms considered in this section are assumed to be onto. Examples of m.p.s.'s are given in Section 3.

2.1. θ-singular $\mathcal{J}$-classes

The first result deals with arbitrary onto morphisms.

Proposition 2.1. *Let $\theta : S \to T$ be an onto morphism, and let J' be a $\mathcal{J}$-class of T. Then $J'\theta^{-1} = J_1 \cup \cdots \cup J_k$ is a union of $\mathcal{J}$-classes of S, and if J_i $(1 \le i \le k)$ is $\le_J$-minimal among $J_1, \ldots, J_k$, then $J_i\theta = J'$. Furthermore, if J' is regular, then the index i is uniquely determined, and J_i is itself regular.*

Proof. Since $s\mathcal{J}s'$ implies $s\theta\mathcal{J}s'\theta$, $J'\theta^{-1} = J_1 \cup \cdots \cup J_k$. Let J_i $(1 \le i \le n)$ be $\le_J$-minimal among $J_1, \ldots, J_k$ and consider $s \in J_i$ and $t \in J'$. Then $s\theta\mathcal{J}t$ and hence $t = (usv)\theta$ for some u and v in S^I. Thus $usv \in J'\theta^{-1}$ and $usv \le_J s$ so that $usv \in J_i$. So, $J_i\theta = J'$.

If J' is regular, let e be an idempotent in J'. Then $e\theta^{-1}$ is a subsemigroup of S and a subset of $J'\theta^{-1}$. So, if J is the $\mathcal{J}$-class of S that contains the minimal ideal of $e\theta^{-1}$, J is a regular $\mathcal{J}$-class of S, and one of $J_1, \ldots, J_k$. Let now $s \in J'\theta^{-1}$. Since $s\theta\mathcal{J}e$, $e = (usv)\theta$ for some u and v in S^I. Then $J \le_J usv \le_J s$. Thus J is the unique $\le_J$-minimal element of $\{J_1, \ldots, J_k\}$. □

Let $\theta : S \to T$ be an m.p.s. A $\mathcal{J}$-class J of S is *θ-singular* if θ is one-to-one on the set $S \setminus J$.

Proposition 2.2. *Let I be an ideal of S, maximal among the ideals of S on which θ is one-to-one. If J is a $\mathcal{J}$-class minimally $\le_J$-above I, then J is θ-singular.*

Proof. Since $I \cup J$ is an ideal strictly containing I, θ is not one-to-one on $I \cup J$. Define the equivalence relation $\sim$ on S by $s \sim s'$ iff either $s = s'$ or $s, s' \in I \cup J$ and $s\theta = s'\theta$. Since $I \cup J$ is an ideal, $\sim$ is a congruence, and θ factorizes as follows:

$$\begin{array}{ccc} S & \xrightarrow{\quad\theta\quad} & T \\ & {\scriptstyle\theta_1}\searrow \quad \nearrow{\scriptstyle\theta_2} & \\ & S/\sim. & \end{array}$$

Since θ is not one-to-one on $I \cup J$, $\sim$ is not the equality and hence θ_1 is not an isomorphism. Therefore θ_2 is an isomorphism: $s\theta = s'\theta$ iff $s \sim s'$. So, whenever $s\theta = s'\theta$ and $s, s' \notin J$, either $s, s' \in I$, on which θ is one-to-one, or $s, s' \in S \setminus (I \cup J)$ where $\sim$ is equality. Thus θ is one-to-one on $S \setminus J$. □

We can deduce from Proposition 2.2 the existence of θ-singular $\mathcal{J}$-classes.

Corollary 2.3. *S contains a θ-singular $\mathcal{J}$-class.*

Proof. Since θ is not one-to-one, an ideal I_0 of S, maximal among the ideals on which θ is one-to-one, exists (recall that $\emptyset$ is an ideal) and is a strict subset of S. Further, ideals are union of $\mathcal{J}$-classes, so that there exist $\mathcal{J}$-classes entirely in $S \setminus I_0$. So we can use Proposition 2.2. $\square$

The existence of θ-singular $\mathcal{J}$-classes implies the following property.

Proposition 2.4. *Let I be an ideal of S on which θ is not one-to-one. Then θ is one-to-one on $S \setminus I$ and $(S \setminus I)\theta \cap I\theta = \emptyset$.*

Proof. Let J be a θ-singular $\mathcal{J}$-class. Since θ is not one-to-one on I, $J \subseteq I$ and hence θ is one-to-one on $S \setminus I \subseteq S \setminus J$. Let then $I_0 = \{s \in I \mid J \nleq_J s\}$: I_0 is an ideal where θ is one-to-one. Let $\sim$ be defined on S by $s \sim s'$ iff either $s = s'$, or $s, s' \in I_0 \cup J$ and $s\theta = s'\theta$. As in the proof of Proposition 2.2, $\sim$ is a congruence and $s\theta = s'\theta$ iff $s \sim s'$. It is a consequence of the definition of $\sim$ that classes of elements of $I_0 \cup J$ and of $S \setminus (I_0 \cup J)$ are disjoint. Thus θ separates $I_0 \cup J$ from $S \setminus (I_0 \cup J)$. $\square$

Let us finally note the following. We denote by S^{r} the *reverse semigroup* of S: $S^{\mathrm{r}} = \{s^{\mathrm{r}} \mid s \in S\}$ and $s^{\mathrm{r}} \cdot t^{\mathrm{r}} = (ts)^{\mathrm{r}}$. If $\theta : S \to T$ is a relational morphism, the *reverse morphism* $\theta^{\mathrm{r}} : S^{\mathrm{r}} \to T^{\mathrm{r}}$ is defined by $s^{\mathrm{r}}\theta^{\mathrm{r}} = \{t^{\mathrm{r}} \mid t \in s\theta\}$ $(s \in S)$. θ is an m.p.s. iff θ^{r} is one, and J, $\mathcal{J}$-class of S, is θ-singular iff J^{r} is θ^{r}-singular.

2.2. Properties of m.p.s.'s

Let $\mathcal{K}$ be one of the Green relations $\mathcal{J}$, $\mathcal{R}$, $\mathcal{L}$ or $\mathcal{H}$, and let $\theta : S \to T$ be an onto morphism. We say that θ is a *$\mathcal{K}$-morphism* if $s\theta\mathcal{K}s'\theta$ implies $s\mathcal{K}s'$. We say that θ is *injective on $\mathcal{K}$-classes* (or a *$\gamma(\mathcal{K})$-morphism*) if $s\theta = s'\theta$ and $s\mathcal{K}s'$ implies $s = s'$ $(s, s' \in S)$.

One can check [10] that $\mathcal{H}$-morphisms are $\mathcal{J}$-morphisms, that $\mathcal{J}$-morphisms are **LG**-morphisms, and that morphisms that are injective on $\mathcal{H}$-classes are aperiodic.

Let $\theta : S \to T$ be an m.p.s. A main result of [5] is the following:

Proposition 2.5 [5]. *θ is either injective on $\mathcal{H}$-classes, or is a $\mathcal{H}$-morphism, and θ cannot be both.*

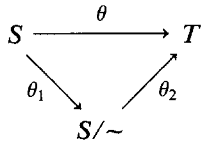
Proof. It is clear that an $\mathcal{H}$-morphism that is injective on $\mathcal{H}$-classes is one-to-one, and hence is not an mp.s.

Let us suppose that θ is not injective on $\mathcal{H}$-classes, and let us define $\sim$ on S by $s \sim s'$ if $s\theta = s'\theta$ and $s\mathcal{H}s'$. Then $\sim$ is not trivial. Let J be a θ-singular $\mathcal{J}$-class: θ being one-to-one on $S \setminus J$, $\sim$ is the equality on $S \setminus J$.

Moreover, $\sim$ is a congruence. Let indeed $s \sim s'$ and $u, v \in S^{\cdot}$. If s or s' is in $S \setminus J$, then s and s' are in $S \setminus J$ since $s\mathcal{H}s'$. Thus $s = s'$ and $usv = us'v$. If $s, s' \in J$, then

$s\theta = s'\theta$ and $(usv)\theta = (us'v)\theta$. If one of usv and $us'v$, say usv, is in J, then, by the classical properties of Green relations $s\mathcal{H}s'$ implies $usv\mathcal{H}us'v$. So $usv \in J$ iff $us'v \in J$, in which case $usv\mathcal{H}us'v$: $usv \sim us'v$. Finally, if both usv and $us'v$ are out of J, $usv = us'v$ since $(usv)\theta = (us'v)\theta$.

But θ factorizes as



and θ_1 is not an isomorphism. So θ_2 is one-to-one and $s\theta = s'\theta$ iff $s \sim s'$. This means that $s\theta = s'\theta$ implies $s\mathcal{H}s'$: θ is a $\mathcal{H}$-morphism. □

We now prove that S contains at most two θ-singular $\mathcal{J}$-classes.

Proposition 2.6. *Let θ be an m.p.s.*

(1) *The number of θ-singular $\mathcal{J}$-classes is* 1 *or* 2.

(2) *If θ is a $\mathcal{J}$-morphism, S contains exactly one θ-singular $\mathcal{J}$-class J, $J\theta$ is a $\mathcal{J}$-class in T and $J\theta\theta^{-1} = J$.*

(3) *If θ is not a $\mathcal{J}$-morphism, there is exactly one $\mathcal{J}$-class J' of T such that $J'\theta^{-1}$ is not a $\mathcal{J}$-class. Then $J'\theta^{-1}$ is the union of two $\mathcal{J}$-classes, $J'\theta^{-1} = J \cup Q$, of which one at least is θ-singular. Further, any θ-singular $\mathcal{J}$-class is either J or Q.*

Proof. (1) is a consequence of (2) and (3).

(2) Let θ be a $\mathcal{J}$-morphism and assume that J_1 and J_2 are distinct θ-singular $\mathcal{J}$-classes. Then θ is one-to-one on $S \setminus (J_1 \cup J_2)$, $J_1 (\subseteq S \setminus J_2)$ and $J_2 (\subseteq S \setminus J_1)$. Moreover, since θ is a $\mathcal{J}$-morphism, $J_1\theta$, $J_2\theta$ and $(S \setminus (J_1 \cup J_2))\theta$ are pairwise disjoint, and hence θ is one-to-one: this is a contradiction.

(3) If θ is not a $\mathcal{J}$-morphism, there exists a $\mathcal{J}$-class J' of T such that $J'\theta^{-1} = J_1 \cup \cdots \cup J_k$, $k \geq 2$. Let J be $\leq_J$-minimal in $\{J_1, \ldots, J_k\}$ and Q be $\leq_J$-minimal in $\{J_1 \cup \cdots \cup J_k\} \setminus \{J\}$. Let also $I = \{s \in S \mid s \leq_J J \text{ or } s \leq_J Q\}$. I is an ideal containing $J \cup Q$ and hence, after Proposition 2.1, θ is not one-to-one on I. Then, by Proposition 2.4, $I\theta \cap (S \setminus I)\theta = \emptyset$. So, if $k \geq 3$, for any J_i that is different from J and Q, $J_i \subseteq S \setminus I$ and hence $J_i\theta \cap I\theta = \emptyset$. But $J_i\theta \subseteq J' = (J \cup Q)\theta \subseteq I\theta$: we have a contradiction and thus $J'\theta^{-1} = J \cup Q$.

Since θ is not one-to-one on $J \cup Q$, no other $\mathcal{J}$-class than J or Q can be θ-singular. This also proves the uniqueness of J'. □

3. Classification of mp.s.'s

In this section, we describe the four classes of m.p.s.'s. These results extend and are more detailed than the results of [5]. (See also [10].)

3.1. Definition of the classes

Let $\theta: S \to T$ be an m.p.s.
- θ is in *class* I if θ is a $\mathscr{H}$-morphism.
- θ is in *class* II if θ is a $\mathscr{J}$-morphism and is injective on $\mathscr{H}$-classes.
- θ is in *class* III if, using the notations of Proposition 2.6(3), $J'\theta^{-1} = J \cup Q$, and $J <_J Q$.
- θ is in *class* IV if, using the notations of Proposition 2.6(3), $J'\theta^{-1} = J \cup Q$, and J and Q are not $\mathscr{J}$-comparable.

After Propositions 2.5 and 2.6, these classes are disjoint and cover the class of all m.p.s.'s. In terms of θ-singular $\mathscr{J}$-classes, we have the following:

Proposition 3.1. *Let θ be an m.p.s.*

(1) *θ is in class* I *or* II *iff there is exactly one θ-singular $\mathscr{J}$-class J, and $J\theta \cap (S \setminus J)\theta = \emptyset$.*

(2) *θ is in class* III *iff there exists a θ-singular $\mathscr{J}$-class Q such that, if J' is the $\mathscr{J}$-class containing $Q\theta$, $J'\theta^{-1} = J \cup Q$ with $J <_J Q$ (J may also be θ-singular).*

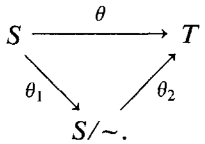
(3) *θ is in class* IV *iff there exist two θ-singular $\mathscr{J}$-classes that are not $\mathscr{J}$-comparable.*

Proof. (1) is a consequence of Proposition 2.6(2) and of the fact that $\mathscr{H}$-morphisms are $\mathscr{J}$-morphisms.

(2) Let θ be in class III. By definition, we have $J'\theta^{-1} = J \cup Q$ and, after Proposition 2.1, $J\theta = J'$. Also, after Proposition 2.6(3), θ is a bijection from $S \setminus (J \cup Q)$ onto $T \setminus J'$. Let $\sim$ be defined on S by $s \sim s'$ iff either $s = s'$ or $s, s' \in J$ and $s\theta = s'\theta$.

$\sim$ is a congruence. Let indeed $s \sim s'$ ($s, s' \in S$) and $u, v \in S^{\cdot}$. If $s = s'$, then $usv = us'v$. Otherwise, $s, s' \in J$ and $s\theta = s'\theta$, and hence $(usv)\theta = (us'v)\theta$. If $(usv)v \in J'$, then usv, $us'v \in J'\theta^{-1} = J \cup Q$ and, since usv, $us'v \leq_J J$, we have usv, $us'v \in J$, so that $usv \sim us'v$.

θ factorizes as



If θ_2 is an isomorphism, then $s\theta = s'\theta$ implies $s \sim s'$ and hence sJs', which is absurd since $Q\theta \subset J' = J\theta$ (by Proposition 2.1). So θ_1 is an isomorphism, i.e. $\sim$ is the equality on S. Thus θ is one-to-one on $S \setminus Q$, and Q is θ-singular.

The converse is immediate.

(3) Let θ be in class IV. By definition, we have $J'\theta^{-1} = J \cup Q$, where J and Q are not $\mathscr{J}$-comparable. After Propositions 2.1 and 2.6(3), $J\theta = Q\theta = J'$ and θ is a bijection from $S \setminus (J \cup Q)$ onto $T \setminus J'$. Let $\sim_Q$ and $\sim_J$ be defined on S by $s \sim_Q s'$ (resp. $s \sim_J s'$) iff either $s = s'$ or $s, s' \in Q$ (resp. J) and $s\theta = s'\theta$.

By the same reasoning as above, we prove that $\sim_J$ is a congruence that is the equality. Thus Q is θ-singular. But, here, J and Q play symmetrical parts, so that J is also θ-singular.

The converse is immediate. □

We now turn to the description of each of the four classes. We shall freely use the results and notations of Section 1.

3.2. *Class* I

Let $\theta: S \to T$ be in class I, and let J be the θ-singular $\mathscr{J}$-class. We shall distinguish two subclasses based on the regularity of J: we say that θ is in *class* I_R (resp. class I_N) if J is a regular (resp. null) $\mathscr{J}$-class. The regularity of a $\mathscr{J}$-class is stable under reverse so that we have:

Proposition 3.2. *θ is in class I_R (resp. I_N, I) iff so is θ^{r}.* □

We have $J^0 = \mathscr{M}^0(A, B, G, P)$ and $J'^0 = \mathscr{M}^0(A', B', G', P')$ for some A, A', B, B', G, G', P, P'. Since θ is a $\mathscr{H}$-morphism, it is also an $\mathscr{R}$-, an $\mathscr{L}$- and a $\mathscr{J}$-morphism. Further, $J\theta = J'$ (Proposition 2.6(2)). In particular, the image of a $\mathscr{R}$- (resp. $\mathscr{L}$-, $\mathscr{H}$-) class of J is a whole $\mathscr{R}$- (resp. $\mathscr{L}$-, $\mathscr{H}$-) class of J'. So we can choose $A' = A$ and $B' = B$ with the mappings α and β equal respectively to the identity functions of A and B. Also the group morphism $\tilde{\theta}: G \to G'$ is onto, i.e. there exists a non-trivial normal subgroup N of G such that $G' = G/N$ (with $\tilde{\theta}$ the canonical projection). It is then easy to check that $(a, g, b)\theta = (a, gN, b)$ for all $a \in A$, $b \in B$, $g \in G$ and that, if $p_{b,a} = 0$, then $p'_{b,a} = 0$ and if $p_{b,a} \neq 0$, then $p'_{b,a} = p_{b,a}N$ $(a \in A,\ b \in B)$.

Conversely, let N' be a non-trivial normal subgroup of G. For any $s \in S \setminus J$, let $w\varphi = s\theta \in T \setminus J'$, and for any $(a, g, b) \in J = A \times G \times B$, let $(a, g, b)\varphi = (a, gN', b) \in \mathscr{M}^0(A, B, G/N', P \cdot N')$. Then there exists a semigroup structure on $V = (T \setminus J') \cup \mathscr{M}^0(A, B, G/N', P \cdot N')$ that makes $\varphi: S \to V$ a morphism iff N' satisfies the following conditions:

(C1) For any $s \in S \setminus J$, $a \in A$, $b \in B$, $g_1, g_2 \in G$ such that $g_1N' = g_2N'$, either $s(a, g_1, b) = s(a, g_2, b)$, or $s(a, g_1, g) = (a', g'_1, b)$, $s(a, g_2, b) = (a', g'_2, b)$ and $g'_1N' = g'_2N'$.

(C2) For any $s \in S \setminus J$, $a \in A$, $b \in B$, $g_1, g_2 \in G$ such that $g_1N' = g_2N'$, either $(a, g_1, b)s = (a, g_2, b)s$, or $(a, g_1, b)s = (a, g'_1, b')$, $(a, g_2, b)s = (a, g'_2, b')$ and $g'_1N' = g'_2N'$.

(C3) If $p_{b,a} = 0$, $g_1N' = g'_1N'$, $g_2N' = g'_2N'$, then $(a_1, g_1, b)(a, g_2, b_2) = (a_1, g'_1, b) \cdot (a, g'_2, b_2)$.

Clearly, then, φ is onto, one-to-one on $S \setminus J$ and an $\mathscr{H}$-morphism. Further, if $N = N'$, then $V = T$ and $\varphi = \theta$, and hence N satisfies (C1–C3).

We now prove that φ is an m.p.s. iff N' is minimal (for the inclusion relation) among all non-trivial normal subgroups of G, which implies by an elementary well-known theorem of group theory that N' is isomorphic to $T \times \cdots \times T$ for T a simple group. Let us suppose first that N' is a minimal normal subgroup of G and satisfies

(C1–C3). If $\varphi=\varphi_1\varphi_2$, then $\tilde{\varphi}: G\to G/N'$ factorizes into $\tilde{\varphi}=\tilde{\varphi}_1\tilde{\varphi}_2$. Since N' is minimal, one of $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ is an isomorphism and hence, either φ_1 or φ_2 is an isomorphism: φ is an m.p.s.

Conversely, let us assume that N' is a non-trivial normal subgroup satisfying (C1–C3) and such that φ is an m.p.s. Let N'' be a normal subgroup of G contained in N'. If N'' satisfies (C1–C3) as well, let φ_1 be the induced morphism, defined as above: $s\varphi_1=s\varphi$ if $s\notin J$, $(a,g,b)\varphi_1=(a,gN'',b)$. $s\varphi_1=s'\varphi_1$ $(s,s'\in S)$ implies $s\varphi=s'\varphi$ so that $\varphi=\varphi_1\varphi_2$. By minimality of φ, one of $\varphi_1\varphi_2$ is an isomorphism, and hence either $N''=\{1\}$ or $N''=N'$. Thus N' is a minimal normal subgroup.

We conclude by showing that if $N''\subset N'$, N'' always satisfies (C1–C3). Let us notice first that $(gN'')_{g\in G}$ is a partition of G that refines $(gN')_{g\in G}$. The verification of (C3) is then immediate. We now prove that N'' satisfies (C1) (the verification of (C2) is dual). Let $s\in S\setminus J$ and $g_1,g_2\in G$ such that $g_1N''=g_2N''$. Then $g_1N'=g_2N'$. Since N' satisfies (C1), for all $a\in A$ and $b\in B$, either $s(a,g_1,b)=s(a,g_2,b)$, or $s(a,g_1,b)=(a',g_1',b)$, $s(a,g_2,b)=(a',g_2',b)$ and $g_1'N'=g_2'N'$. In this last case, recall that, with the notations of Subsection 1.1,

$$s(a,g_1,b)=su_a(h_0\cdot g_1)v_b=u_{a'}(h_0\cdot g_1')v_b=u_{a'}(\bar{u}_{a'}su_a)(h_0\cdot g_1)v_b.$$

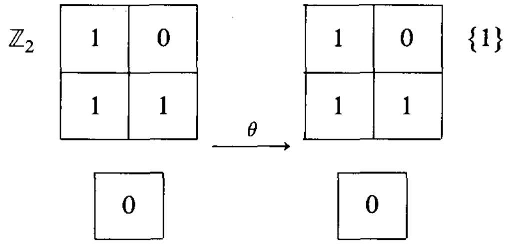
The left translation by $\bar{u}_{a'}su_a$ preserves $H^S_{1,1}$: let g_s be the element of G (acting on the right on $H^S_{1,1}$) that induces the same permutation of $H^S_{1,1}$. Then $g_1'=g_sg_1$ and, similarly, $g_2'=g_sg_2$. So $g_1N''=g_2N''$ implies $g_1'N''=g_2'N''$.

So we have proved

Proposition 3.3. *Let θ be a morphism. θ is an m.p.s. in class* I *with θ-singular $\mathcal{J}$-class $J=\mathcal{M}^0(A,B,G,P)\setminus\{1\}$ iff there exists a minimal non-trivial normal subgroup N of G such that θ is a bijection from $S\setminus J$ onto $T\setminus J'\theta$ and $(a,g,b)\theta=(a,gN,b)$ for all $(a,g,b)\in J$. Hence $N\cong S\times\cdots\times S$ for some simple group S.* □

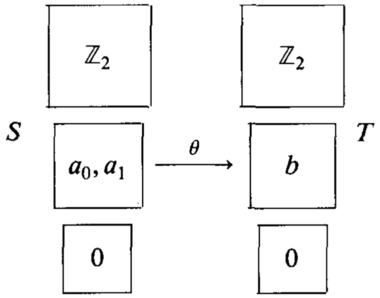
Example 1. (0) The typical example of I_R is $G\to G/N$, N a minimal normal subgroup of G a group.

(1) Let $S=\mathcal{M}^0(2,2,\mathbb{Z}_2,\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right))$, $T=\mathcal{M}^0(2,2,\{1\},\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right))$ and $\theta: S\to T$ be given by $(a,g,b)\theta=(a,1,b)$ for all a,b in $\{1,2\}$, g in $\mathbb{Z}_2$: θ is an m.p.s. of class I_R.



(2) Let now $S=\{1,c,a_0,a_1,0\}$ be given by $c^2=1$, $ca_i=a_ic=a_{1-i}$ and $a_ia_j=0$ $(i,j\in\{0,1\})$. Let $T=\{1,c,b,0\}$ be given by $c^2=1$, $cb=bc=b$, $b^2=0$, and let

$\theta: S \to T$ be defined by $1\theta = 1$, $c\theta = c$, $a_0\theta = a_1\theta = b$, $0\theta = 0$. Then θ is an m.p.s. of class I_N.



3.3. Class II

Let $\theta: S \to T$ be an m.p.s. of class II. We know that θ is a $\mathcal{J}$-morphism that is injective on $\mathcal{H}$-classes, that it has exactly one θ-singular $\mathcal{J}$-class J, that $J' = J\theta$ is a $\mathcal{J}$-class of T and that θ is one-to-one from $S \setminus J$ onto $T \setminus J'$. We shall sometimes identify the sets $S \setminus J$ and $T \setminus J'$ and consider that θ is the identity function on that set.

Proposition 3.4. *One can find representations of J and J', $J^0 = \mathcal{M}^0(A, B, G, P)$ and $J'^0 = \mathcal{M}^0(A', B', G', P')$ such that one of the following conditions holds*:

(1) $B = B'$; $G = G'$; *there exists an onto mapping* $\alpha: A \to A'$ *such that* $(a, g, b)\theta = (a\alpha, g, b)$ *and* $p_{b,a} = p'_{b,a\alpha}$ *for all* $a \in A$, $b \in B$ *and* $g \in G$.

(2) $A = A'$; $G = G'$; *there exists an onto mapping* $\beta: B \to B'$ *such that* $(a, g, b)\theta = (a, g, b\beta)$ *and* $p_{b,a} = p'_{b\beta,a}$ *for all* $a \in A$, $b \in B$ *and* $g \in G$.

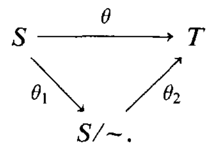
Note that a morphism that satisfies both (1) and (2) is an isomorphism and hence not an m.p.s. If an m.p.s. θ satisfies (1), we say that it *identifies rows* (or is in *class* $\mathrm{II}_{\mathrm{row}}$). We say that it *identifies columns* (or is in class $\mathrm{II}_{\mathrm{col}}$) if it satisfies (2). It is immediate that

Proposition 3.5. *Class* II *is preserved under the passage from θ to θ^{r}, but classes* $\mathrm{II}_{\mathrm{row}}$ *and* $\mathrm{II}_{\mathrm{col}}$ *are interchanged.* □

Proof of Proposition 3.4. Let $J^0 = \mathcal{M}^0(A, B, G, P)$ and $J'^0 = \mathcal{M}^0(A', B', G', P')$ be arbitrary representations of J and J'. Since $J\theta = J'$ and θ is injective on $\mathcal{H}$-classes, the mappings α and β (with the notations of Subsection 1.1) are onto, and θ is a monomorphism.

Note first that, since θ is not a $\mathcal{H}$-morphism, one at least of α and β is not one-to-one. Let now $\sim$ be defined on S by $s \sim s'$ iff either $s = s'$, or $s, s' \in J$, $s\mathcal{R}s'$ and $s\theta = s'\theta$. $\sim$ is a congruence. Let indeed $s \sim s'$ and $u, v \in S^{\cdot}$. Then $(usv)\theta = (us'v)\theta$. If $(usv)\theta \notin J'$, then $usv = us'v$ since θ is one-to-one on $S \setminus J$. If $(usv)\theta \in J'$, then both usv and $us'v$ lie in $J = J'\theta^{-1}$. Since $s\mathcal{R}s'$, we have $us\mathcal{R}us'$ and hence $usv\mathcal{R}us'v$. So $usv \sim us'v$.

If β is not one-to-one, then $\sim$ is not trivial. Indeed there exist s_1, s_2 in J such that $s_1 \mathscr{R} s_2$, $s_1\theta \mathscr{H} s_2\theta$. Then $s_1\theta = s_2\theta u$ for some $u \in T^{\cdot}$ and, if $v\theta = u$, $s_1\theta = (s_2 v)\theta$ and $s_1 \mathscr{R} s_2 v$ (since $s_2 v$ lies necessarily in $J = J'\theta^{-1}$). But θ factorizes as



So θ_2 is an isomorphism, i.e. $s\theta = s'\theta$ implies $s\mathscr{R}s'$. Similarly, if α is not one-to-one, $s\theta = s'\theta$ implies $s\mathscr{L}s'$. So, if neither α nor β is one-to-one, $s\theta = s'\theta$ implies $s\mathscr{H}s'$ and, since θ is injective on $\mathscr{H}$-classes, θ is one-to-one. This is a contradiction and hence, one of α and β, say α, is one-to-one. We can then assume that $A = A'$ and $\alpha = \mathrm{id}_A$.

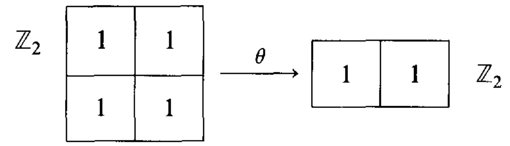
We now prove that θ is an isomorphism from G onto G' by showing that θ maps some $\mathscr{H}$-class of J onto a $\mathscr{H}$-class of J'. Recall that $h_0 \in H^S_{1,1}$ and $h'_0 = h_0\theta \in H^T_{1,1}$. Then $k = uh'_0$ for some $u \in T^{\cdot}$ and, if $v\theta = u$, $(vh_0)\theta = k$. Thus $vh_0 \in J = J'\theta^{-1}$, $vh_0 \mathscr{L} h_0$ and, since α is one-to-one, $vh_0 \mathscr{R} h_0$. So $vh_0 \in H^S_{1,1}$ and $(vh_0)\theta = k$.

We can now assume that $G = G'$ and $\theta = \mathrm{id}_G$. Recall that the isomorphism of J'^0 with $\mathscr{M}^0(A, B', G, P')$ is given by the choice of elements $u'_a, \overline{u'_a}, v'_b, \overline{v'_b}$ $(a \in A,\ b \in B')$ of $T^{\cdot}$. We shall construct $u_a, \bar{u}_a, v_b, \bar{v}_b$ $(a \in A,\ b \in B)$, i.e. an isomorphism of J^0 with $\mathscr{M}^0(A, B, G, P)$, such that $(a, g, b)\theta = (a, g, b\beta)$ $(a \in A,\ b \in B,\ g \in G)$. Let $b \in B$ and $b' = b\beta \in B'$. Since θ is a bijection from $R^S_1 \cap L^S_b$ onto $R^T_1 \cap L^T_{b'}$, there exists $k_b \in R^S_1 \cap L^S_b$ such that $k_b\theta = h'_0 v_{b\beta}$ and there exists $v_b \in S^{\cdot}$ such that $k_b = h_0 v_b$. Let also $\bar{v}_b$ be such that $h_0 = k_b \bar{v}_b$. Similarly, for $a \in A$, there exists $k_a \in R^S_a \cap L^S_1$ such that $k_a\theta = u'_a h'_0$. We can then choose u_a and $\bar{u}_a$ in $S^{\cdot}$ such that $k_a = u_a h_0$ and $h_0 = \bar{u}_a k_a$. Then the computation of Subsection 1.1 turns into a simpler form and we have $(a, g, b)\theta = (a, g, b\beta)$. It is clear, then, that $p_{b,a} = p'_{b\beta,a}$. $\square$

Example 2. (1) The most typical example of Class II is 'identify equal rows' or 'proportional rows', e.g. let

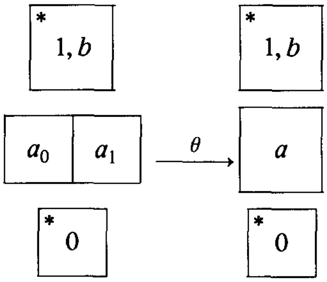
$$S = \mathscr{M}^0\left(2, 2, \mathbb{Z}_2, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right), \qquad T = \mathscr{M}^0(1, 2, \mathbb{Z}_2, (1 \quad 1))$$

and $\theta : S \to T$ be given by $(a, z, b)\theta = (1, z, b)$ for all a, b in $\{1, 2\}$, z in $\mathbb{Z}_2$: θ is an m.p.s. of class $\mathrm{II}_{\mathrm{row}}$ and $\theta^{\mathrm{r}} : S^{\mathrm{r}} = S^{\mathrm{r}} \to T^{\mathrm{r}} = \mathscr{M}^0(2, 1, \mathbb{Z}_2, \binom{1}{1})$ is in class $\mathrm{II}_{\mathrm{col}}$.



(2) For $k \geq 1$, let $U_k = \{1, a_1, \ldots, a_k\}$ be the monoid given by $a_i a_j = a_j$ $(1 \leq i, j \leq k)$. Let $k \geq 1$ and $\theta : U_{k+1} \to U_k$ be given by $1\theta = 1$, $a_i\theta = a_i$ $(1 \leq i \leq k)$ and $a_{k+1}\theta = a_k$. Then θ is in class $\mathrm{II}_{\mathrm{col}}$.

(3) Note that the θ-singular $\mathcal{J}$-class of a class II-m.p.s. need not be regular. Let $S=\{1,b,a_0,a_1,0\}$ with $b^2=1$, $a_ia_j=0$, $ba_i=a_i$ and $a_ib=a_{1-i}$ $(0\leq i,j\leq 1)$, and let $T=\{1,b,a,0\}$ with $b^2=1$, $ab=ba=a$ and $a^2=0$. If $\theta:S\to T$ is given by $1\theta=1$, $0\theta=0$, $b\theta=b$ and $a_i\theta=a$ $(i=0,1)$, then θ is in class $\mathrm{II}_{\mathrm{col}}$ and $\{a_0,a_1\}$ is the θ-singular $\mathcal{J}$-class.



3.4. Class III

Let $\theta:S\to T$ be an m.p.s. Recall that θ is in class III if there exists a $\mathcal{J}$-class J' of T such that $J'\theta^{-1}$ is the union of two $\mathcal{J}$-classes J and Q such that $J<_J Q$ and Q is θ-singular. So θ induces a bijection from J onto J' (Proposition 2.1) and hence, J^0 and J'^0 are isomorphic 0-simple semigroups. Also, since θ is not a $\mathcal{H}$-morphism, θ is injective on $\mathcal{H}$-classes (Proposition 2.5) and hence the Schützenberger group of Q is a subgroup of the Schützenberger group of J.

It will be of some use to distinguish within class III three subclasses based on the regularity of J', J and Q. If J' is a null $\mathcal{J}$-class, then so are J and Q. In this case, we say that θ is in *class* $\mathrm{III}_{N>N}$ (the letter N stands for null). Else, J' is regular and so is J, after Proposition 2.1. If Q is null we say that θ is in *class* $\mathrm{III}_{N>R}$, and if Q is regular, we say that θ is in *class* $\mathrm{III}_{R>R}$. Note that there is no class $\mathrm{III}_{R>N}$.

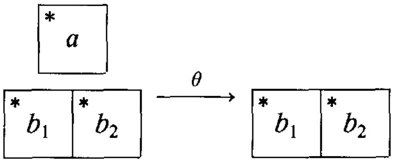
The following proposition is easy to check:

Proposition 3.6. *θ is in class* III (*resp.* $\mathrm{III}_{N>N}$, $\mathrm{III}_{N>R}$, $\mathrm{III}_{R>R}$) *iff θ^r is in this class.* □

We now give examples of m.p.s.'s in these classes, with either one or two θ-singular $\mathcal{J}$-classes.

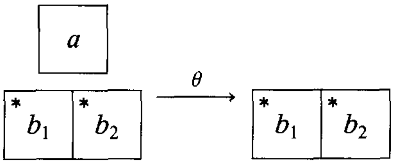
Example 3. (0) The typical example of $\mathrm{III}_{R>R}$ is $S\to\{1\}=T$ where S is a regular Rees matrix semigroup over $\{1\}$, i.e. $S=\mathcal{M}^0(A,B,\{1\},C)$ where C is a matrix of zeros and ones with distinct rows and distinct columns and no zero rows or zero columns. There is only one θ-singular $\mathcal{J}$-class unless $C=(1)$, i.e. let $S=U_1=\{1,0\}$, $T=\{1\}$ and θ be the morphism from S onto T. Then $Q=\{1\}$ and $J=\{0\}$ are θ-singular, and θ is in class $\mathrm{III}_{R>R}$.

(1) Let $S=\{a,b_1,b_2\}$ and $T=\{b_1,b_2\}$ be given by $a^2=a$, $b_ib_j=b_j$, $ab_j=b_i$ and $b_ia=b_2$ $(1\leq i,j\leq 2)$. Let $\theta:S\to T$ be given by $b_i\theta=b_i$ $(i=1,2)$ and $a\theta=b_2$. Then θ is again in class $\mathrm{III}_{R>R}$ and $Q=\{a\}$ and $J=\{b_1,b_2\}$ are θ-singular J-classes.

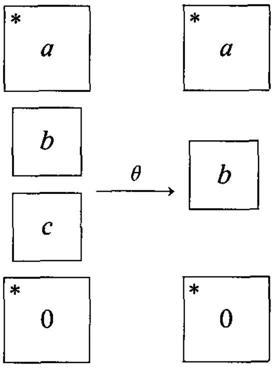


(2) Let $S=\{a,0\}$ be given by $a^2=0$, $T=\{1\}$ and θ be the morphism from S onto T. θ is in class $\mathrm{III}_{N>R}$ and both $Q=\{a\}$ and $J=\{0\}$ are θ-singular.

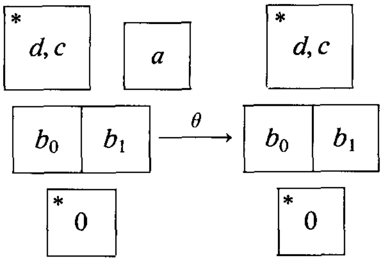
Let now $S=\{a,b_1,b_2\}$ and $T=\{b_1,b_2\}$ be given by $b_ib_j=b_j$, $a^2=b_ia=b_1$ and $ab_i=b_i$ $(1\le i,j\le 2)$. Let $\theta: S\to T$ be given by $b_i\theta=b_i$ $(1\le i\le 2)$ and $a\theta=b_1$. Then θ is again in class $\mathrm{III}_{N>R}$ and $Q=\{a\}$ and $J=\{b_1,b_2\}$ are the θ-singular $\mathcal{J}$-classes.



(3) Let $S=\{0,a,b,c\}$ and $T=\{0,a,b\}$ be given by $a^2=a$, $ab=ba=ac=ca=c$, $bc=cb=b^2=c^2=0$, and let $\theta: S\to T$ be defined by $a\theta=a$, $b\theta=c\theta=b$, $0\theta=0$. θ is in class $\mathrm{III}_{N>N}$ and both $Q=\{b\}$ and $J=\{c\}$ are θ-singular.



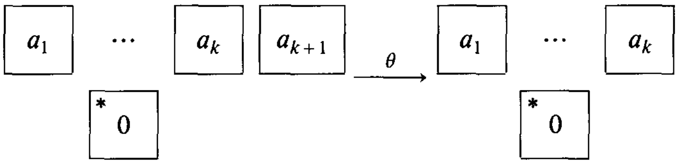
Let finally $S=\{a,b_0,b_1,c,d,0\}$ and $T=\{b_0,b_1,c,d,0\}$ be given by $d^2=c^2=d$, $cd=dc=c$ (so $\{c,d\}\approx\mathbb{Z}_2$), $b_ib_j=ab_i=b_ia=0$, $b_i=db_i=b_i$, $b_ic=cb_i=b_{1-i}$ $(0\le i,j\le 1)$, $ad=da=b_1$ and $ca=ac=b_0$. Let also $\theta: S\to T$ be given by the identity on T and $a\theta=b_1$. θ is again in class $\mathrm{III}_{N>N}$ and $Q=\{a\}$ is the only θ-singular $\mathcal{J}$-class.



3.5. Class IV

Recall that for an m.p.s. $\theta: S \to T$ in class IV, there exists a unique $\mathcal{J}$-class J' of T such that $J'\theta^{-1}$ is the union of two $\mathcal{J}$-classes J and Q, that are not $\mathcal{J}$-comparable. After Propositions 2.1, 2.5 and 3.1, θ induces a bijection, respectively from J and Q onto J', θ is injective on $\mathcal{H}$-classes, and both J and Q are θ-singular. Also, J', J and Q are null $\mathcal{J}$-classes. Also J'^0, J^0 and Q^0 are isomorphic and J', J and Q have isomorphic sets of $\mathcal{R}$- and $\mathcal{L}$-classes and isomorphic Schützenberger groups.

Example 4. If $k \geq 1$, let $S_k = \{0, a_1, \ldots, a_k\}$ with $a_i a_j = 0$. Then $\theta: S_{k+1} \to S_k$ given by $0\theta = 0$, $a_i\theta = a_i$ $(1 \leq i \leq k)$ and $a_{k+1}\theta = a_k$ is an m.p.s. of class IV.



3.6. V-morphisms and m.p.s.'s

Using the characterizations of aperiodic and **LG**-morphisms given in Proposition 1.1 and the description of m.p.s.'s in the above sections, it is easy to check the following:

Proposition 3.7. *Let θ be an m.p.s. θ is an aperiodic morphism iff θ is not in class* I_R. *θ is an* **LG**-*morphism iff θ is not in class* $\mathrm{III}_{R>R}$. □

Note that these properties of morphisms are rigid, that is, if $\theta_1: S \to V$ and $\theta_2: V \to T$ are onto morphisms, $\theta = \theta_1\theta_2$ is aperiodic (resp. an **LG**-morphism, an **LI**-morphism, a regular morphism) iff so are θ_1 and θ_2 (Propositions 1.1 and 1.2). Thus we have

Proposition 3.8. *Let θ be any onto morphism, and let $\theta = \theta_1 \cdots \theta_n$ be a factorization of θ in m.p.s.'s. Then θ is aperiodic (resp. an* **LG**-*morphism, an* **LI**-*morphism) iff none of the θ_i's is in class* I_R *(resp.* $\mathrm{III}_{R>R}$, I_R*) nor* $\mathrm{III}_{R>R}$. □

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